

New Upper and Lower Bounds on the Rado Numbers

William Gasarch, Russel Moriarty

Department of Computer Science,
The University of Maryland at College Park

Nithin Tamma

Port Huron Northern High School

Abstract

If \mathcal{E} is a linear homogenous equation and $c \in \mathbb{N}$ then the Rado number $R_c(\mathcal{E})$ is the least N so that any c -coloring of the positive integers from 1 to N contains a monochromatic solution. Rado characterized for which \mathcal{E} $R_c(\mathcal{E})$ always exists. The original proof of Rado's theorem gave enormous bounds on $R_c(\mathcal{E})$ (when it existed). In this paper we establish better upper bounds, and some lower bounds, for $R_c(\mathcal{E})$ for some c and \mathcal{E} . In the appendix we use some of our theorems, and ideas from a probabilistic SAT solver, to find many new Rado Numbers.

1 Introduction

Notation 1.1. If $n \in \mathbb{N}$ then $[n]$ is the set $\{1, \dots, n\}$.

Let \mathcal{E} be a linear homogenous equation and $N \in \mathbb{N}$. If you color c -color $[N]$ you may or may not get a monochromatic solution to \mathcal{E} .

Definition 1.2. If \mathcal{E} is a linear homogenous equation.

1. Let $c \in \mathbb{N}$. $R_c(\mathcal{E})$ is the least positive integer N (if it exists) such that any c -coloring of $[N]$ contains a monochromatic solution to \mathcal{E} . If we do not include the subscript then it is assumed to be 2.
2. \mathcal{E} is c -regular if $R_c(\mathcal{E})$ exists.
3. \mathcal{E} is regular if, for all c , \mathcal{E} is c -regular.

In 1916 Schur proved that the equation $x+y=z$ is regular [11]. In 1933 Rado, a graduate student of Schur's, determined exactly which systems of equations are regular [9]. His proof used an extension of van der Waerden's theorem and hence lead to large bounds on $R_c(\mathcal{E})$. We present his theorem for single equations:

Theorem 1.3 (Rado's Theorem). *The equation $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = 0$ is regular if and only if there exists a subset $I \in [n]$ such that $\sum_{i \in I} a_i = 0$.*

For a proof of Rado's theorem consult the books by Graham, Rothchild, and Spencer [14], Landman and Robertson [4] or the free on-line book of Gasarch, Kruskal, Parrish [1].

It is an open problem in Ramsey Theory to find better upper bounds on the Rado Numbers. To date 2-color Rado numbers have only been determined for a few classes of equations. In this paper we prove several theorems that give much better upper bounds on $R_c(\mathcal{E})$ for several c and \mathcal{E} . We also have some computational results in the appendix.

Previous results have determined the Rado numbers for some classes of equations, completely characterized the 2-color Rado numbers for equations of the form $a(x + y) = bz$ [7], while Robertson and Myers gave results and conjectures for four variable equations of the form of $x + y + kz = jw$ [5]. Here we examine the case $a(x - y) = bz$ and $x + ay = abz$, bounding the two color Rado numbers of both. Additionally some four variable equations are considered, and their Rado numbers proven. Furthermore, a result of Rado is extended, showing that all non-trivial two variable equations are not 2-regular. A new proof of Rado's single equation theorem is given, providing better bounds on Rado numbers in certain cases. Additionally, a probabilistic method approach is provided which gives lower bounds on Rado numbers of equations with an arbitrary number of colors. We conclude with our algorithm for computing Rado numbers, tables of computed 2 and 3-color Rado numbers, and conjectures that follow.

2 Summary of Results

We list our main results:

1. Results on 2-coloring

- (a) $R_2(x - y = bz) = b^2 + 3b + 1$
- (b) $R_2(a(x - y) = bz) = a^2$, $a > b$
- (c) $R_2(a(x - y) = bz) \geq b^2 + b + 1$, $a \leq b$
- (d) $R_2(x + ay = abz) \geq a^2$
- (e) $R_2(px + a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = abz) \geq a^2$ for p relatively prime to a and $a_i \equiv 0 \pmod{a}$.
- (f) $R_2(x + ay = 2az) = a^2$
- (g) $R_2(x + y + az = (a + 1)w) = 5$, $a > 3$
- (h) $R_2(2x + 2y + az = (a + 3)w) = 10$, $a > 24$
- (i) $R_2(3x + 3y + az = (a + 5)w) = 15$, $a \geq 30$
- (j) $R_2(ax = bz) = \infty$, $a \neq b$

2. Results on c -coloring

- (a) $R_c(\mathcal{E}) \leq m^2 + 3m + 1$, where \mathcal{E} is an equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ that includes $I \subset [1, n]$ where $\sum_{i \in I} a_i = 0$ and a $q \in I$ such that q divides $\sum_{i \notin I} a_i$.
- (b) $R_c(x - y = az) > \frac{\sqrt{(a+c-1)^2 + 8c^2(a^2+a)} - a - c + 1}{2}$
- (c) $R_c(a(x - y) = bz) \geq \frac{bc^3}{c(b+3)} + \frac{2b+3}{b+3}$

3 Bounds on 2-Color Rado Numbers

We obtain new upper and lower bounds of 2-color Rado numbers for several classes of equations. We also obtain a new proof of Rado's single equation theorem in the $c = 2$ case which leads to better upper bounds for some Rado Numbers.

3.1 2-Color Rado Numbers for $a(x - y) = bz$.

In this section we characterize the Rado numbers of equations of the form $a(x - y) = bz$.

Theorem 3.1. $R(x - y = bz) = b^2 + 3b + 1$

Lower Bound. We show a coloring of $[1, b^2 + 3b]$ lacking a monochromatic solution to $x - y = bz$. Consider the coloring defined by $RRRR \dots$ (Total of b R's) followed by $BBBB \dots$ (Total of $b^2 + b$ B's) followed by $RRRR \dots$ (Total of b R's). It is shown that this coloring does not admit a monochromatic solution to $x - y = bz$. If z is R , and in $[b^2 + 2b + 1, b^2 + 3b]$, $bz \geq b^3 + 2b^2 + b$. But $x - y \leq b^2 + 3b - 1 < bz$ for $b > 1$. So, if z is R it must be in $[1, b]$. If x is in $[b^2 + 2b + 1, b^2 + 3b]$ and y is in $[1, b]$, $x - y \geq b^2 + 2b > bz$ because $bz \leq b^2$. On the other hand, if x and y are in $[1, b]$, $x - y \leq b - 1 < bz$ because $bz \geq b$. If z is B , $x - y \leq b^2 + b - 1 < bz$, because $bz \geq b^2 + b$. So, there can be no monochromatic solution of $x - y = bz$ under this coloring of $[1, b^2 + 3b]$. \square

Upper Bound. We show that for all colorings COL there must exist a monochromatic solution to $x - y = bz$ in $[1, b^2 + 3b + 1]$. We notice that $(q + bd, q, d)$ is a solution to $x - y = bz$ for any $q, d \in \mathbb{N}$. Note that $COL(q) \neq COL(q + qb)$ due to the solution $(q + qb, q, q)$. We can assume that $1 \in R$, so $(b + 1) \in B$. Then, $b^2 + 2b + 1 \in R$. Because $(b^2 + 3b + 1, b^2 + 2b + 1, 1)$ is a solution, $b^2 + 3b + 1 \in B$.

Case: $2 \in R$. $b + 2 \in B$, and because $(b^2 + 3b + 1, b + 1, b + 2)$ is a solution, $b + 2 \in R$, giving a contradiction. So $2 \in B$

Case: $3 \in R$. Because $(3b + 1, b + 1, 2)$ is a solution, $3b + 1 \in R$. But, since $(3b + 1, 1, 3)$ is a solution, $3b + 1 \in B$, giving a contradiction. Thus, $3 \in B$.

$(3b + 1, b + 1, 2)$ is a solution, so $3b + 1 \in R$. Because $(4b + 1, b + 1, 3)$ is also a solution, $4b + 1 \in R$. But $(4b + 1, 3b + 1, 1)$ is a solution, so $4b + 1 \in B$, giving a contradiction. So any coloring of $[1, b^2 + 3b + 1]$, with $b \in \mathbb{N}$ must contain a monochromatic solution to $x - y = az$. \square

Theorem 3.2. $R(a(x - y) = bz) = a^2$ for $a > b$.

Lower Bound. We show a coloring of $[1, a^2 - 1]$ lacking a monochromatic coloring to $a(x - y) = bz$ for $a > b$. Consider the coloring defined by

$$\chi(x) = \begin{cases} 1 & \text{when } x \equiv 0 \pmod{a} \\ 0 & \text{otherwise} \end{cases}$$

We may assume that a and b are relatively prime by dividing common factors as necessary. Taking the equation mod a we see that $bz \equiv 0 \pmod{a}$. So, $z \equiv 0 \pmod{a}$. For a monochromatic solution under $\chi(x)$ to exist, $x, y \equiv 0 \pmod{a}$. Let $z = na$, with $n \geq 1$. Then, rewrite the equation as

$$a(x - y) = bna$$

Divide the equation by a to get

$$x - y = nb$$

Reduce mod a to see that $nb \equiv 0 \pmod{a}$. Because a and b are relatively prime, $n \equiv 0 \pmod{a}$. Since $n \geq 1$, $n = ma$ with $m \geq 1$. Then $z = na = ma^2 \geq a^2$. Thus, no monochromatic solution to $a(x - y) = bz$ exists under $\chi(x)$ in $[1, a^2 - 1]$. \square

Upper Bound. We show for all colorings COL of $[1, a^2]$ there must exist a monochromatic coloring to $a(x - y) = bz$ with $a > b$. Assume, for contradiction, that there exists a coloring of $[1, a^2]$ without a monochromatic solution to $a(x - y) = bz$. Without loss of generality, let $a \in R$. Considering the solution $(a, a - b, a)$, we see that $a - b \in B$. From $(a + b, a, a)$, $a + b \in B$ as well. Then $2a \in R$ so $a + b, a - b, 2a$ is not monochromatic. We see that from $(2a, 2(a - b), 2a)$, $2(a - b) \in B$. From $(2a + b, 2a, a)$, we have that $2a + b \in B$. Then from $(2a + b, 2(a - b), 3a)$ we see that $3a \in R$. Continuing in this fashion, we see that $na \in R$ for $1 \leq n \leq a$. But $(a^2, (a - b)a, a^2)$ is a monochromatic solution, a contradiction. \square

Theorem 3.3. $R(a(x - y) = bz) \geq b^2 + b + 1$ for $a \leq b$.

Lower Bound. We show a coloring of $[1, b^2 + b]$ lacking a monochromatic coloring to $a(x - y) = bz$ for $a \leq b$. Consider the coloring defined by

$$\chi_1(x) = \begin{cases} 1 & \text{when } 1 \leq x \leq ab \text{ AND } x \equiv 0 \pmod{a} \\ 0 & \text{when } x \not\equiv 0 \pmod{a} \\ 0 & \text{when } ab + 1 \leq x \leq b^2 + b \end{cases}$$

We may assume that a is relatively prime to b by dividing common factors as necessary. Now, reducing the equation mod a we see that $bz \equiv 0 \pmod{a}$. So, $z \equiv 0 \pmod{a}$, and we can let $z = na$, with $n \geq 1$. First we show that $COL(z) \neq 1$. If $COL(z) = 1$, then $x - y \leq a(b - 1)$. But then $x - y = ma$ with $1 \leq m \leq b - 1$. However, $a(x - y) = ma^2$, and because ma^2 is not divisible by b , $a(x - y)$ cannot equal bz . So $COL(z) = 0$ and $z \geq a(b + 1)$. For $a(x - y) = bz$ to hold, $x - y \geq b^2 + b$, but $x - y \leq b^2 + b - 1$. So, there does not exist a monochromatic solution to $a(x - y) = bz$ under $\chi_1(x)$. \square

Based on computed 2-color Rado numbers we make the following conjecture:

Conjecture 3.4. $R(a(x - y) = bz) = b^2 + b + 1$ for $a \leq b$.

3.2 Rado Numbers for $x + ay = abz$.

We give results for equations of the form $x + ay = abz$, proving a general lower bound applicable to equations with arbitrarily many variables. At the end of the section a few conjectures are posed based on evidence from computed 2 and 3-color Rado.

Theorem 3.5. $R(px + a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = abz) \geq a^2$ for p relatively prime to a and $a_i \equiv 0 \pmod{a}$.

Proof. Consider the coloring $\chi(x)$ defined by

$$\chi(x) = \begin{cases} 1 & \text{when } x \equiv 0 \pmod{a} \\ 0 & \text{otherwise} \end{cases}$$

Reduce the equation mod a to get $px \equiv 0 \pmod{a}$. For a monochromatic solution under $\chi(x)$, x_i, z must be $\equiv 0 \pmod{a}$. Because p is relatively prime to a , $x \equiv 0 \pmod{a}$. Let $x = na$, and rewrite the equation as

$$pna + a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = abz$$

Let $\frac{a_i}{a} = c_i$ and divide the equation by a to get

$$pn + c_1x_1 + c_2x_2 + \cdots + c_nx_n = bz.$$

Reduce this mod a to find $pn \equiv 0 \pmod{a}$. Because p is relatively prime to a , $n \equiv 0 \pmod{a}$. So $n = ma$, with $m \geq 1$, and $x = na = ma^2$. Thus, $x \geq a^2$ and a monochromatic solution to $px + a_1x_1 + a_2x_2 + a_3x_3 + \cdots = abz$ cannot exist under $\chi(x)$ in $[1, a^2 - 1]$. \square

Theorem 3.6. $R(x + ay = 2az) = a^2$.

Lower Bound. The lower bound is given by the theorem above. \square

Upper Bound. We show that any coloring COL of $[1, a^2]$ contains a monochromatic solution to $x + ay = 2az$. We show that if $a \equiv 1 \pmod{2}$, we are guaranteed a monochromatic solution in $[1, a^2]$. Because (a^2, a, a) is a solution to the above equation, $COL(a^2) \neq COL(a)$. From the solution $(a, a, 1)$, we see that $COL(a) \neq COL(1)$, implying that $COL(a^2) = COL(1)$. $(a^2, a^2, \frac{a^2+a}{2})$ is another solution to the equation, so $COL(a^2) \neq COL(\frac{a^2+a}{2})$. Because $(\frac{a^2+a}{2}, \frac{a+1}{2}, \frac{a+1}{2})$ is also a solution to the equation, $COL(\frac{a^2+a}{2}) \neq COL(\frac{a+1}{2})$, implying that $COL(a^2) = COL(\frac{a+1}{2})$. However, we now have a monochromatic solution: $(a^2, 1, \frac{a+1}{2})$. For the case $a \equiv 0 \pmod{2}$, we see that we can write a as $m2^i$ where $m \equiv 1 \pmod{2}$. Then, the equation becomes $x = m2^iy = m2^{i+1}z$. Reducing mod 2^i , we see that $x \equiv 0 \pmod{2^i}$. So we can divide the equation by 2^i , giving $x + my = 2mz$. This is equivalent to the case of $a \equiv 0 \pmod{2}$, which was proved above. \square

Using a similar argument, we find a lower bound for the 2-color Rado numbers of equations of the form $x + a^ny = a^nbz$, with $a, b \in \mathbb{N}$.

From computed values of 2-color Rado numbers, we present the following conjecture.

Conjecture 3.7. $R(x + ay = abz) = a^2$ with $a \geq 2b - 1$.

3.3 Some 2-Color Rado Numbers for $b(x + y) + az = (a + 2b - 1)w$.

In this section we give some 2-color Rado numbers for some four variable equations.

Theorem 3.8. $R(x + y + az = (a + 1)w) = 5$ for $a > 3$.

Lower Bound. It is easy to check that the coloring $RBRB$ contains no monochromatic solutions to $x + y + az = (a + 1)w$ in $[1, 4]$. \square

Upper Bound. Assume, for contradiction, that there exists a coloring of $[1, 5]$ without a monochromatic solution to $x + y + az = (a + 1)w$. Without loss of generality let $1 \in R$. From the solution $(1, 1, 2, 2)$, we see that $2 \in B$. From $(2, 2, 4, 4)$ we see that $4 \in R$. Then, $3 \in B$ because of the solution $(3, 1, 4, 4)$, and $5 \in R$ from $(2, 3, 5, 5)$. But then we have the monochromatic solution $(4, 1, 5, 5)$, a contradiction. \square

Theorem 3.9. $R(2x + 2y + az = (a + 3)w) = 10$ for $a > 24$

Lower Bound. It is easy to see that the coloring $RBBRBBRRBR$ does not contain a monochromatic solution to $2x + 2y + az = (a + 3)w$ in $[1, 9]$. \square

Upper Bound. Assume, for contradiction, that there exists a coloring of $[1, 10]$ that does not contain a monochromatic solution to $2x + 2y + az = (a + 3)w$. Without loss of generality let $1 \in R$. From $(1, 2, 2, 2)$ we see that $2 \in B$. From $(2, 4, 4, 4)$, $4 \in R$. $5 \in B$ so $(1, 5, 4, 4)$ is not monochromatic. From $(3, 3, 4, 4)$ we see that $3 \in B$, and from $(3, 6, 6, 6)$, $6 \in R$. $8 \in B$ so $(4, 8, 8, 8)$ is not monochromatic, and $7 \in R$ because of the solution $(5, 7, 8, 8)$. We see that $10 \in R$ from $(2, 10, 8, 8)$ and that $9 \in B$ from $(9, 6, 10, 10)$. But we have the monochromatic solution $(9, 3, 8, 8)$, a contradiction. \square

Theorem 3.10. $R(3x + 3y + az = (a + 5)w) = 15$ for $a \geq 30$.

Lower Bound. It is easy to see that the coloring $RBRBBRBBRRBRBRB$ admits no monochromatic solutions to $3x + 3y + az = (a + 5)w$ in $[1, 14]$. \square

As the Upper Bound is very similar to the previous cases, it is left to the reader.

It is tempting to attempt to extend the results to the general case $b(x + y) + az = (a + 2b - 1)z$, but we quickly see that these do not seem to follow the $R_2(\mathcal{E}) = 5b$ that seems to hold for $1 \leq b \leq 3$.

3.4 When is an Equation 2-Regular?

Here we expand on a result of Rado with respect to the 2-regularity of linear equations. Recall that an equation is k -regular if there exists an $n \in \mathbb{N}$ such that all k -colorings of $[1, n]$ contain a monochromatic solution. Rado proved the following result [9]. The following proof was included in [15], we include it for completeness.

Theorem 3.11. *All equations in three or more variables with positive and negative coefficients are 2-regular.*

Proof. Let $\sum_{i=1}^k \alpha_i x_i = \sum_{i=1}^\ell \beta_i y_i$ be our equation, where $k \geq 2$, $\ell \geq 1$, $\alpha_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$, and $\beta_i \in \mathbb{Z}^+$ for $1 \leq i \leq \ell$. By setting $x = x_1 = x_2 = \dots = x_{k-1}$, $y = x_k$, and $z = y_1 = y_2 = \dots = y_\ell$, we may consider solutions to

$$ax + by = cz,$$

where $a = \sum_{i=1}^{k-1} \alpha_i$, $b = \alpha_k$, and $c = \sum_{i=1}^\ell \beta_i$. We will denote $ax + by = cz$ by \mathcal{E} .

Let $m = \text{lcm}\left(\frac{a}{\gcd(a,b)}, \frac{c}{\gcd(b,c)}\right)$. Let (x_0, y_0, z_0) be the solution to \mathcal{E} with $\max(x, y, z)$ a minimum, where the maximum is taken over all solutions of positive integers to \mathcal{E} . Let $A = \max(x_0, y_0, z_0)$.

Assume, for a contradiction, that there exists a 2-coloring of \mathbb{Z}^+ with no monochromatic solution to \mathcal{E} . First, note that for any $n \in \mathbb{Z}^+$, the set $\{in : i = 1, 2, \dots, A\}$ cannot be monochromatic, for otherwise $x = x_0n$, $y = y_0n$, and $z = z_0n$ is a monochromatic solution, a contradiction.

Let $x = m$ so that $\frac{bx}{a}, \frac{bx}{c} \in \mathbb{Z}^+$. Letting red and blue be our two colors, we may assume, without loss of generality, that x is red. Let y be the smallest number in $\{im : i = 1, 2, \dots, A\}$ that is blue. Say $y = \ell m$ so that $2 \leq \ell \leq A$.

For some $n \in \mathbb{Z}^+$, we have that $z = \frac{b}{a}(y - x)n$ is blue, otherwise $\{i\frac{b}{a}(y - x) : i = 1, 2, \dots\}$ would be red, admitting a monochromatic solution to \mathcal{E} . Then $w = \frac{a}{c}z + \frac{b}{c}y$ must be red, for otherwise $az + by = cw$ and z, y , and w are all blue, a contradiction. Since x and w are both red, we have that $q = \frac{c}{a}w - \frac{b}{a}x = \frac{b}{a}(y - x)(n + 1)$ must be blue, for otherwise x, w , and q give a red solution to \mathcal{E} . As a consequence, we see that $\{i\frac{b}{a}(y - x) : i = n, n + 1, \dots\}$ is monochromatic. This gives us that $\{i\frac{b}{a}(y - x)n : i = 1, 2, \dots, A\}$ is monochromatic, a contradiction. □

For another proof consult [12]. Here we show that all two variable equations of the form $ax = by$ are not 2-regular, excluding the trivial case $a = b$.

Theorem 3.12. *The equation $ax = by$ is not 2-regular for $a \neq b$.*

Proof. The case $a = b$ is regular for all k , due to the trivial solution (k, k) . Without loss of generality assume that $b > a$. We may assume that a and b are relatively prime by dividing common factors as necessary. Now, clearly a and b must either both be positive or negative, else no solution (x, y) can exist with $x, y \in \mathbb{N}$. We show a coloring $\chi(x)$ of \mathbb{N} that does not contain a monochromatic solution to $ax = by$. Note that for all $j \in \mathbb{N}$, $(\frac{b}{a})^i < j < (\frac{b}{a})^{i+1}$ for a unique value of i . Now let $v(j) = i$. Define the coloring

$$\chi(x) = \begin{cases} 1 & \text{if } v(x) \equiv 0 \pmod{2} \\ 0 & \text{if } v(x) \equiv 1 \pmod{2} \end{cases}$$

Writing $ax = by$ as $x = \frac{b}{a}y$ we see that, for a particular solution (x, y) , if $v(y) = k$ then $v(x) = k + 1$. Under the coloring $\chi(x)$ there can be no monochromatic solutions to $ax = by$. □

It is interesting to note that the above theorem is valid for all of \mathbb{Q} , as opposed to the other theorems in this paper, in which we are only concerned with \mathbb{N} .

4 Better Bounds on the Rado Function

We present an alternative proof of the single equation Rado's theorem which yields better upper bounds in some cases.

Recall Van der Waerden's theorem:

Theorem 4.1 (Van der Waerden's Theorem). *For all $k, c \in \mathbb{N}$, there exists $W = W(k, c)$ such that for all c -colorings $\chi : [W] \rightarrow [c]$ there exist $a, d \in \mathbb{N}$ such that $\chi(a) = \chi(a + d) = \chi(a + 2d) = \dots = \chi(a + (k - 1)d)$.*

This was first proven by van der Warden [10]. See the books by Graham, Rothchild, and Spencer [3], Landman and Robertson [4] or the free on-line book of Gasarch, Kruskal, Parrish [1] for the proof in English.

This proof gives enormous upper bounds on the numbers $W(k, c)$ that are not primitive recursive. Shelah [8] gave an alternative proof that yields primitive recursive upper bounds. All of the proofs noted above are elementary. Gowers [2] provided a non-elementary proof that yields much better better, though still huge, bounds.

The following variant of Van der Waerden's Theorem is used to prove Rado's theorem and determine better bounds on Rado numbers.

Theorem 4.2 (VDW Variant). *For all $k, l, m, c \in \mathbb{N}$, there exists $U = U(k, l, m, c)$ such that for all c -colorings $\chi : [U] \rightarrow [c]$ there exist $a, d \in \mathbb{N}$ such that*

$$\chi(a) = \chi(a + md) = \chi(a + 2md) = \dots = \chi(a + (k - 1)md) = \chi(ld)$$

Proof. The proof is by induction on c . Clearly for all k, l, m , we have that $U(k, l, m, 1) = \max\{1 + (k - 1)m, l\}$. For the induction step we assume $U(k, l, m, c - 1)$ exists and use it to prove the existence of $U(k, l, m, c)$. Let χ be a c -coloring of $[W(k', c)]$, where $k' = (k - 1)lmU(k, l, m, c - 1) + 1$. By the definition of $W(k', c)$, there exist a, d such that

$$\chi(a) = \chi(a + d) = \chi(a + 2d) = \dots = \chi(a + (k' - 1)d)$$

Without loss of generality assume this color is RED. This implies that for all $i \in [U(k, l, m, c - 1)]$,

$$\chi(a) = \chi(a + mid) = \chi(a + 2mid) = \dots = \chi(a + (k - 1)mid) = \text{RED}$$

Now there are two cases:

CASE 1: There exists $i \in [U(k, l, m, c - 1)]$ such that $\chi(lid) = \text{RED}$. Therefore

$$\chi(a) = \chi(a + mid) = \chi(a + 2mid) = \dots = \chi(a + (k - 1)mid) = \chi(lid) = \text{RED}$$

and we are done with this case.

CASE 2: For all $i \in [U(k, l, m, c - 1)]$, $\chi(lid) \neq \text{RED}$. This implies that χ gives a $(c - 1)$ -coloring of $\{ild\}_{i \in [U(k, l, m, c - 1)]}$. By the definition of $U(k, l, m, c - 1)$, there exist a', d' such that

$$\chi(a'ld) = \chi((a' + md')ld) = \chi((a' + 2md')ld) = \dots = \chi((a' + (k - 1)md')ld) = \chi(ld'ld)$$

Substituting $A = a'ld$ and $D = d'ld$ gives

$$\chi(A) = \chi(A + mD) = \chi(A + 2mD) = \dots = \chi(A + (k - 1)mD) = \chi(lD)$$

and we are done. □

4.1 Proof of Rado's Theorem

We begin by presenting the lemma used in our proof of Rado's Theorem, then present Rado's Theorem itself.

Lemma 4.3. *For all $c, l \in \mathbb{N}$ and for all $m \neq 0 \in \mathbb{Z}$, there exists a $P = P(l, m, c) \in \mathbb{N}$ such that for all c -colorings of $[P]$, there exists $a, d \in \mathbb{N}$ such that $a, ld, a + md \in [P]$ are monochromatic.*

Proof. Let U be the function from Theorem 4.2. Let $P = U(2, l, m, c)$ and $\chi : [P] \rightarrow [c]$ be a coloring of $[P]$. By the definition of U , there exist a, d such that $\chi(a) = \chi(a + md) = \chi(ld)$, which is exactly what we wanted to prove. \square

We use the following notation for the Rado number in the remainder of this section:

Definition 4.4. *The Rado number $R(a_1, \dots, a_n; c)$ is the smallest R such that for all c -colorings of $[1, R]$ there exists a monochromatic solution to $a_1x_1 + \dots + a_nx_n = 0$.*

Theorem 4.5 (Rado's Theorem). *For all $a_1, \dots, a_n \in \mathbb{Z}$, if there exists an $I \subseteq [1, n]$ such that $\sum_{i \in I} a_i = 0$, then for all $c \geq 1$, $\exists R(a_1, \dots, a_n; c)$ such that for all c -colorings of $[1, R]$ there exists a monochromatic solution to $a_1x_1 + \dots + a_nx_n = 0$, where each $x_i \in [1, R]$. $R(a_1, \dots, a_n; c)$ satisfies the following upper bound:*

$$R(a_1, a_2, \dots, a_n; c) \leq P \left(\frac{LCM(\sum_{i \notin I} a_i, a_q)}{\sum_{i \notin I} a_i}, -\frac{LCM(\sum_{i \notin I} a_i, a_q)}{a_q}, c \right)$$

Proof. Define

$$s = \sum_{i \notin I} a_i$$

Choose $q \in I$ such that $|LCM(s, a_q)|$ is minimal, and let $u = LCM(s, a_q)$, where u is chosen to have the same sign as s . We claim that if there exist positive integers a and d such that $a, \frac{ud}{s}, a - \frac{ud}{a_q} \in [R]$ are monochromatic, then there exists a monochromatic solution to the above equation. Namely,

$$x_i = \begin{cases} a - \frac{ud}{a_q} & \text{if } i = q \\ a & \text{if } i \neq q \in I \\ \frac{ud}{s} & \text{if } i \notin I \end{cases}$$

We can verify this as follows:

$$\begin{aligned} \sum_{i=1}^n a_i x_i &= \sum_{i \in I} a_i x_i + \sum_{i \notin I} a_i x_i \\ &= \sum_{i \in I} a_i a - a_q \frac{ud}{b} + \sum_{i \notin I} a_i \frac{ud}{s} \\ &= 0 - a_q \frac{ud}{a_q} + s \frac{ud}{s} \\ &= 0 \end{aligned}$$

We apply Lemma 1 with $l = \frac{u}{s}$ and $m = -\frac{u}{a_q}$ to obtain an R large enough to guarantee the existence of a monochromatic triple $a, ld, a + md \in [R]$. Since u was chosen to have the same sign as s , l is guaranteed to be positive in our application of Lemma 1. If a_q also has the same sign as s then $m < 0$, whereas if a_q and s have opposite signs then $m > 0$.

Formally, we have shown the following:

$$\begin{aligned} R(a_1, \dots, a_n; c) &\leq P\left(\frac{u}{s}, -\frac{u}{a_q}, c\right) \\ &= P\left(\frac{LCM(s, a_q)}{s}, -\frac{LCM(s, a_q)}{a_q}, c\right) \\ &= P\left(\frac{LCM(\sum_{i \notin I} a_i, a_q)}{\sum_{i \notin I} a_i}, -\frac{LCM(\sum_{i \notin I} a_i, a_q)}{a_q}, c\right) \end{aligned}$$

□

The VDW proof of Lemma 1 gives the following upper bound on Rado numbers:

$$\begin{aligned} R(a_1, \dots, a_n; c) &\leq P\left(\frac{LCM(\sum_{i \notin I} a_i, a_q)}{\sum_{i \notin I} a_i}, -\frac{LCM(\sum_{i \notin I} a_i, a_q)}{a_q}, c\right) \\ &= P\left(\frac{u}{s}, -\frac{u}{a_q}, c\right) \end{aligned}$$

4.2 Quadratic Upper Bound on $R_2(\mathcal{E})$ in a Special Case

This section deals with the class of equations where, after forming $I \subseteq [n]$ such that $\sum_{i \in I} a_i = 0$, there exists a $q \in I$ such that a_q divides $s = \sum_{i \notin I} a_i$. For equations that fall into this category, $u = LCM(s, a_q) = s$. Therefore in the application of Lemma 1, $l = \frac{u}{s} = 1$ and $m = \frac{u}{a_q}$. In addition we restrict our attention to the 2-color case.

Lemma 4.6 ($c = 2, l = 1$). *For all $m \in \mathbb{N}$ and for all 2-colorings of $[1 + 3m + m^2]$ there exists a monochromatic triple $a, d, a + md \in [R]$.*

Proof. Our general approach is to do a case analysis of the potential colors that small numbers can take. There are two rules we use in this analysis. The first rule comes from taking $d = a$ in the above lemma.

Rule 1. *For any $a \in \mathbb{N}$, $a + am$ cannot be the same color as a , otherwise we are done. This is because $a, a, a + am$ would be a valid triple.*

The second rule is the more general case.

Rule 2. *For any $a, d \in \mathbb{N}$ that share a color, neither $a + md$ nor $d + ma$ can be that same color, otherwise we are done.*

Without loss of generality assume 1 is colored RED. By Rule 1 that means $1 + m$ must be BLUE, which means $(1 + m)^2 = 1 + 2m + m^2$ must be RED. Applying Rule 2 we get that $(1 + 2m + m^2) + m(1) = 1 + 3m + m^2$ must be BLUE.

CASE 1: 2 is RED. By Rule 2 that means $2 + m$ must be BLUE, which by Rule 2 means $(1 + m) + m(2 + m) = 1 + 3m + m^2$ must be RED. Since $1 + 3m + m^2$ must be either RED or BLUE, we are done.

CASE 2A: 2 is BLUE, 3 is RED. Since 2 is BLUE we can apply Rule 2 to it and $1 + m$ to conclude that $(1 + m) + m(2) = 1 + 3m$ is RED. However since 1 and 3 are RED, Rule 2 implies that $1 + 3m$ is BLUE, so we are done.

CASE 2B: 2 is BLUE, 3 is BLUE. Rule 2 implies $(1 + m) + m(2) = 1 + 3m$ and $(1 + m) + m(3) = 1 + 4m$ must be RED, but applying Rule 2 to 1 and $1 + 3m$ implies $1 + 4m$ must be BLUE, so we are done with this case. The result follows from the fact that $1 + 3m + m^2$ is greater than or equal to both $1 + 3m$ and $1 + 4m$ for $m \geq 1$. \square

The next result follows directly from the above lemma.

Theorem 4.7. $R_c(\mathcal{E}) \leq m^2 + 3m + 1$, where \mathcal{E} is an equation $a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$ that includes $I \subset [1, n]$ where $\sum_{i \in I} a_i = 0$ and a $q \in I$ such that q divides $\sum_{i \notin I} a_i$.

5 Lower Bounds on Rado Numbers with the Probabilistic Method

Here we present a new method of bounding Rado numbers, utilizing a probabilistic proof. With this approach, it is possible to obtain lower bounds of Rado numbers in arbitrarily many colors. To the best of our knowledge this is the first case of expressions for bounds of $R_c(\mathcal{E})$ with $c > 2$.

Let us define a few functions that will be used extensively throughout this section.

Definition 5.1. Let \mathcal{E} be an equation in j variables, then $\psi_{\mathcal{E},i}(N)$ be the number of solutions to \mathcal{E} in $[1, N]$, (x_1, x_2, \dots, x_j) , with exactly i distinct x_k .

Definition 5.2. Given an equation \mathcal{E} in j variables, let $\psi_N(\mathcal{E})$, expressed as a function of N , give the number of integral solutions to \mathcal{E} in $[1, N]$.

Clearly, the following theorem holds.

Theorem 5.3. $\psi_{\mathcal{E}}(N) = \sum_{i=1}^j \psi_{\mathcal{E},i}(N)$

We consider the following method: Given an equation \mathcal{E} in j variables, let $\psi_{\mathcal{E},i}(N)$ and $\psi_{\mathcal{E}}(N)$ be defined as above. For each solution $X = (x_1, x_2, \dots, x_j)$, clearly, there must exist at least one pair i, j such that $x_i \neq x_j$. Otherwise, the trivial solution $(1, 1, \dots, 1)$ would be a solution and $R_c(\mathcal{E}) = 1$.

Randomly assign each element of the interval $[1, N]$ to an element of $[1, c]$. For any solution X , let E_X be the event that X is monochromatic under this random coloring. Let $Pr(E)$ denote the probability of event E occurring. Note the following trivial bounds on $Pr(E_X)$:

Theorem 5.4.

$$\frac{1}{c^{j-1}} \leq Pr(E_X) \leq \frac{1}{c}.$$

Proof. Let $X = (x_1, x_2, \dots, x_j)$. As before, we discount the trivial solution $x_l = x_k$ for all $l, k \in [1, j]$. Now, for define i as the number of distinct $x_k \in X$. Clearly, $2 \leq i \leq j$. The probability of X being monochromatic is $\frac{c}{c^i} = \frac{1}{c^{i-1}}$. The bounds on i give the result. \square

Let i be the number of distinct $x_k \in X$, then note that the proof of the above theorem gives

$$Pr(E_X) = \frac{1}{c^{i-1}}.$$

With $\psi_{\mathcal{E},i}(N)$ and $\psi_{\mathcal{E}}(N)$ as defined above, we turn back to the random coloring of $[1, N]$. Let E_s be the event that the coloring contains a monochromatic solution. We aim to show that $Pr(E_s) < 1$, implying the existence of a coloring of $[1, N]$ lacking a monochromatic solution. Let $Pr(E_i)$ be the probability that a randomly selected solution from the set of all solutions to \mathcal{E} in the interval $[1, N]$ contains i distinct x_k . It is not hard to see that $Pr(E_s)$ is simply:

$$Pr(E_s) = \sum_{i=1}^j Pr(E_i) \frac{1}{c^{i-1}}$$

Now,

$$Pr(E_i) = \frac{\psi_{\mathcal{E},i}(N)}{\psi_{\mathcal{E}}(N)}$$

so we have

$$Pr(E_s) = \sum_{i=1}^j \frac{\psi_{\mathcal{E},i}(N)}{\psi_{\mathcal{E}}(N) c^{i-1}}.$$

Recall that $Pr(E_s) < 1$ implies the existence of a coloring of $[1, N]$ without a solution to equation \mathcal{E} . Thus, we have the following theorem.

Theorem 5.5. *Given an equation \mathcal{E} in j variables, $\psi_{\mathcal{E},i}(N), \psi_{\mathcal{E}}(N)$, and c , $R_c(\mathcal{E}) > N$, where N satisfies*

$$\sum_{i=1}^j \frac{\psi_{\mathcal{E},i}(N)}{\psi_{\mathcal{E}}(N) c^{i-1}} < 1.$$

or the equivalent:

$$\sum_{i=1}^j \psi_{\mathcal{E},i}(N) c^{N-i+1} < c^N.$$

We give an example of this method's application to the equation $x - y = az$.

Theorem 5.6. *$R_c(x - y = bz) > N$, where N satisfies*

$$\frac{N(c-1)}{b+1} + \frac{N(N+b)}{2b} < c^2.$$

Proof. It is fairly easy to see that

$$\psi_{\mathcal{E}}(N) = \frac{b(k)(k+1)}{2},$$

with $k = \lfloor \frac{N}{b} \rfloor$. Similarly, we have

$$\psi_{\mathcal{E},2}(N) = \left\lfloor \frac{N}{b+1} \right\rfloor, \psi_{\mathcal{E},3}(N) = \psi_{\mathcal{E}}(N) - \left\lfloor \frac{N}{b+1} \right\rfloor.$$

Direct application of the method outlined above, using

$$\sum_{i=1}^j \psi_{\mathcal{E},i}(N) c^{N-i+1} < c^N.$$

From $\frac{N}{k} \geq \lfloor \frac{N}{k} \rfloor$, we have

$$\frac{b(\frac{N}{b})(\frac{N}{b}+1)}{2} = \frac{N(N+b)}{2b}.$$

Then, after substituting,

$$\sum_{i=1}^j \psi_{\mathcal{E},i}(N) c^{N-i+1} \leq c^{N-1} \frac{N}{b+1} + c^{N-2} \frac{N(N+b)}{2b} - c^{N-2} \frac{N}{b+1}.$$

So, after simplifying, if

$$\frac{N(c-1)}{b+1} + \frac{N(N+b)}{2b} < c^2,$$

by Theorem 5.5, we have the result. \square

Corollary 5.7.

$$R_c(x-y=bz) > \frac{\sqrt{(b+c-1)^2 + 8c^2(b^2+b)} - b - c + 1}{2}.$$

Proof. This follows directly from applying the quadratic formula to the above theorem. \square

Note that the use of this method is only dependent upon finding the functions $\psi_{N,i}(\mathcal{E})$ for families of equations. In individual equations, this method has the potential to be extremely versatile, as the number of solutions to a particular equation in $[1, n]$ in many cases is a simple computation.

We present the following results on $\psi_{\mathcal{E},i}(N)$ for certain families of equations, whose proofs are left to the reader. Note that $\psi_{\mathcal{E},1}(N) = 0$ for $a(x-y) = bz$ and $x+ay = abz$.

Theorem 5.8. Let $\mathcal{E} = a(x-y) = bz$,

$$\psi_{\mathcal{E}}(N) = \sum_{i=b+1}^N \left\lfloor \frac{i-1}{b} \right\rfloor,$$

$$\psi_{\mathcal{E},2}(N) = \left\lfloor \frac{N}{a+b} \right\rfloor + \left\lfloor \frac{N}{a} \right\rfloor, \psi_{\mathcal{E},3}(N) = \sum_{i=b+1}^N \left\lfloor \frac{i-1}{b} \right\rfloor - \psi_{\mathcal{E},2}(N).$$

Corollary 5.9.

$$R_c(a(x - y) = bz) > \frac{b(2a + b)(1 - c)}{a(a + b)} + b\sqrt{(c - 1)^2 \frac{(2a + b)^2}{a(a + b)} + \frac{2}{b}(\frac{b + 3}{2} + c^2)}$$

Proof. This follows from

$$\psi_{\mathcal{E}}(N) = \sum_{i=b+1}^N \left\lfloor \frac{i-1}{b} \right\rfloor \leq \sum_{i=b+1}^N \frac{i-1}{b} = N - b - 2 + \frac{(N - b - 1)(N - b)}{2b},$$

$$\psi_{\mathcal{E},2} = \left\lfloor \frac{N}{a+b} \right\rfloor + \left\lfloor \frac{N}{a} \right\rfloor \leq \frac{N}{a+b} + \frac{N}{a},$$

$$\psi_{\mathcal{E},3} = \sum_{i=b+1}^N \left\lfloor \frac{i-1}{b} \right\rfloor - \psi_{\mathcal{E},2}(N) \leq N - b - 2 + \frac{N - b - 1}{2b} - \frac{N}{a+b} - \frac{N}{a}$$

and application of the quadratic formula to the expression generated by application of Theorem 5.4. \square

We will obtain better lower bounds on $R_c(a(x - y) = bz)$, by using the Lovasz Local Lemma, in Theorem 5.19.

Note that it is not necessary to find the exact value of $\psi_{\mathcal{E},i}(N)$, upper bounds will suffice for application of the method.

Lemma 5.10. *Let $\mathcal{E} = x + ay = abz$, with $a, b \geq 2$,*

$$\psi_{\mathcal{E}}(N) < \left\lfloor \frac{N-1}{b} \right\rfloor \left\lfloor \frac{N}{a} \right\rfloor,$$

$$\psi_{\mathcal{E},2}(N) = \begin{cases} \left\lfloor \frac{N}{a(b-1)} \right\rfloor + \left\lfloor \frac{N}{ab-1} \right\rfloor + \left\lfloor \frac{N(a+1)}{ab} \right\rfloor & \text{when } b \equiv 0 \pmod{a+1} \\ \left\lfloor \frac{N}{a(b-1)} \right\rfloor + \left\lfloor \frac{N}{ab-1} \right\rfloor + \left\lfloor \frac{N}{ab} \right\rfloor & \text{otherwise} \end{cases}$$

$$\psi_{\mathcal{E},3}(N) < \left\lfloor \frac{N-1}{b} \right\rfloor \left\lfloor \frac{N}{a} \right\rfloor - \psi_{\mathcal{E},2}.$$

This bound can be used to find bounds on the Rado numbers of equations of the form $x + ay = abz$ with the method presented above.

Now, we find the corresponding $\psi_{\mathcal{E}}(N)$ of an equation in an arbitrary number of variables.

Theorem 5.11. *Let $\mathcal{E} = x_1 + x_2 + \dots x_j = x_{j+1} + x_{j+2} + \dots x_k$,*

$$\psi_{\mathcal{E}}(N) = \sum_{i=j}^N \binom{i-1}{j-1} \binom{i-1}{j-k-1}.$$

Proof. Assume $j \geq k$, the alternative case is equivalent by symmetry. From a combinatorial argument, we have that the number of solutions in positive integers to $x_1 + x_2 + \dots + x_j = m$ is $\binom{m-1}{j-1}$. Similarly, the number of solutions to $x_{j+1} + x_{j+2} + \dots + x_k = m$ is $\binom{m-1}{j-k-1}$. So, the number of solutions to $x_1 + x_2 + \dots + x_j = x_{j+1} + x_{j+2} + \dots + x_k$ is $\sum_{i=j}^N \binom{i-1}{j-1} \binom{i-1}{j-k-1}$. \square

From this result, we can find bounds on the Rado numbers of equations of the form of \mathcal{E} in arbitrary number of colors and variables, as long as we can find the corresponding $\psi_{\mathcal{E},i}(N)$ for each equation.

Note that using this method to bound Rado numbers depends upon finding a closed form expression for the functions $\psi(N)_{\mathcal{E},i}$, which may become a difficult problem. However, given N and equation \mathcal{E} , computing the number of solutions to \mathcal{E} in the interval $[1, N]$ is not a difficult computation. Additionally, determining the number of distinct values contained within each solution is relatively simple. Thus, we present a simple algorithmic approach to our method outlined above that can be used to bound Rado numbers.

Briefly, the algorithm computes the values of $\psi_{\mathcal{E},i}(N)$ and $\psi_{\mathcal{E},i}(N+1)$ by counting the number of distinct x_k in each solution in $[1, N]$ and $[1, N+1]$ respectively. If

$$\sum_{i=1}^j \psi_{\mathcal{E},1}(n)c^{n-i+1} < c^n, \text{ and } \sum_{i=1}^j \psi_{\mathcal{E},1}(n+1)c^{n-i+2} \geq c^{n+1},$$

N must be the maximal integer that satisfies the inequality, so $R_c(\mathcal{E}) > N$.

```

Input:  $E = a_1x_1 + a_2x_2 + \dots + a_jx_j = 0$  AND  $c$ 
Set  $n = k$ 
while TRUE
    Find Solutions in  $[1, n]$  and  $[1, n+1]$ 
    Count Solutions with  $i$  distinct  $x_k$ ; Assign values to  $\psi_{\mathcal{E},i}(N)$ ,  $\psi_{\mathcal{E},i}(N+1)$ 
    if  $\sum_{i=1}^j \psi_{\mathcal{E},1}(n)c^{n-i+1} < c^n$  AND  $\sum_{i=1}^j \psi_{\mathcal{E},1}(n+1)c^{n-i+2} \geq c^{n+1}$ 
        Return  $n$ 
    else Increment  $n$ 

```

5.1 Using the Lovász Local Lemma

Note that overcounting, by assuming that each solution to \mathcal{E} is independent of the others, will make the bounds from this method fairly loose. To deal with the dependence amongst solutions sets, we will utilize the concept of a dependency graph and a theorem known as the Lovász Local Lemma, used extensively in the probabilistic method.

We use the following definition of a dependency graph for an equation \mathcal{E} in the interval $[1, N]$.

Definition 5.12 (Dependency graph). *Given an equation \mathcal{E} in j variables with l solutions in $[1, N]$, let X_i represent the i th j -tuple that satisfies \mathcal{E} . The dependency graph G on the solution sets X_i is constructed as follows: for every X_k if $X_k \cap X_n \neq \emptyset$ for $n \in [1, N]$, $(k, n) \in E(G)$.*

The following theorem is a sieve method used in instances where many of the events in a probability space are independent, but there does exist some dependency between distinct events. For proof, consult [13].

Theorem 5.13 (Lovász Local Lemma). *Let A_1, A_2, \dots, A_k be events in a probability space Ω such that $\Pr[A_i] \leq p < 1$. Define d_i as the number of events that are pairwise dependent to A_i , for $i \in [1, k]$, and $d = \max(d_i)$. If $ep(d+1) < 1$, there is a probability > 0 that none of the events A_i occur.*

In order to use the Lovász Local Lemma to bound Rado numbers, we need to define some terms

Definition 5.14.

1. v_i is the degree of $G_N(\mathcal{E})$, the dependency graph of equation \mathcal{E} over $[1, N]$.
2. $\phi_N(\mathcal{E})$ is the $\max(v_i)$, the maximum degree of $G_N(\mathcal{E})$. Note that $\phi_N(\mathcal{E}) = d$ in the Lovasz Local Lemma. $\phi_N(\mathcal{E})$ can also be considered as the maximum number of dependent solutions

Note that $\phi_N(\mathcal{E})$ can also be considered as the maximum number of dependent solutions in $[N]$. If we can find $\phi_N(\mathcal{E})$ as a function of N , we can use the Lovasz Local Lemma to determine lower bounds on the c -color Rado number of equation \mathcal{E} . We describe this method in the following theorem.

Theorem 5.15. *Let $\phi_N(\mathcal{E})$ be as defined above, for equation \mathcal{E} in i variables. Then, applying the Lovasz Local Lemma, $R_c(\mathcal{E}) > N$, where N satisfies*

$$\phi_N(\mathcal{E}) + 1 < \frac{c^i}{e}.$$

This follows from the obvious fact that a random c -coloring of an i -tuple will be monochromatic with probability $\frac{1}{c^i}$.

Following, we give $\phi_N(\mathcal{E})$ of some classes of equations. The proofs are fairly simple, and are left to the reader.

Lemma 5.16. *Let $a < b$ and let \mathcal{E} be $a(x - y) = bz$. Then*

$$\phi_N(\mathcal{E}) = \left\lfloor \frac{N-1}{b} \right\rfloor + 2 \left\lfloor \frac{N-b-1}{b} \right\rfloor + N - b + 1.$$

Note that for $a < b$, $\phi_N(a(x - y) = bz)$ does not depend upon the value of a .

Lemma 5.17. *Let $E = a(x - y) = bz$, for $a > b$,*

$$\phi_N(\mathcal{E}) = 2 \left\lfloor \frac{N}{a} \right\rfloor + N - b + 1.$$

We show the application of this method by considering the case $b = 2$.

Theorem 5.18.

$$R_c(x - y = 2z) \geq \frac{2c^3}{5e} + \frac{7}{5}.$$

$$R_c(a(x - y) = 2z) \geq \frac{ac^3}{(a+2)e}$$

for $a > 2$.

Proof. For the case $a = 1$,

$$\left\lfloor \frac{N-1}{2} \right\rfloor + 2 \left\lfloor \frac{N-3}{2} \right\rfloor + N - 1 \leq \frac{N-1}{2} + \frac{2(N-3)}{2} + N - 1 = \frac{5N-7}{2} + 1.$$

Letting $d = \frac{5N-7}{2} + 1$, and from $p = \frac{1}{e^3}$, and application of the Lovasz Local Lemma, we have the result.

For the case $a > 2$,

$$2 \left\lfloor \frac{N}{a} \right\rfloor + N - 1 \leq \frac{2N}{a} + N - 1 = \frac{(a+2)N}{a} - 1.$$

Letting this equal d , and $p = \frac{1}{e^3}$ as above, and applying the Lovasz Local Lemma, we have the result. \square

The proof of the following is similar.

Theorem 5.19. For $a < b$,

$$R_c(a(x - y) = bz) \geq \frac{bc^3}{e(b+3)} + \frac{2b+3}{b+3}.$$

The lower bound on $R_c(\mathcal{E})$ from Theorem 5.14 applies to any equation \mathcal{E} , not just regular ones. Of course, if $R_c(\mathcal{E})$ does not exist then the bound is not useful. Nevertheless, we present a lower bound on $R_c(\mathcal{E})$ for an \mathcal{E} that is not necessarily regular (though it may be c -regular for some values of c).

Theorem 5.20. Let $E = x + ay = abz$,

$$\phi_N(\mathcal{E}) = 3 \left\lfloor \frac{N+1}{b} \right\rfloor.$$

Note that determining $\psi_N(\mathcal{E})$ is a much easier task than $\phi_N(\mathcal{E})$, but utilizing the Lovasz Local Lemma gives much better bounds on $R_c(\mathcal{E})$. We believe that refining these arguments, and defining $\psi_N(\mathcal{E})$ and $\phi_N(\mathcal{E})$ for many more classes of equations, will help us better understand the behavior of $R_c(\mathcal{E})$ for $c > 2$.

6 Conjectures on Rado Numbers

Using empirical results from Rado numbers computed using our algorithm, we propose the following conjectures:

6.1 $x + qy = q^2z$

Here we provide a conjecture on Rado numbers of equations of the form $x + qy = q^2z$ with $q \in \mathbb{N}$.

Conjecture 6.1.

$$R(x + qy = q^2z) = \begin{cases} \frac{q^3}{2} & \text{if } q \equiv 0 \pmod{2} \\ \frac{q^3+q}{2} & \text{if } q \equiv 1 \pmod{2} \end{cases}$$

6.2 4-Variable Conjecture

Here, a conjecture is posed regarding the extension of the three variable equation: $x + ay = 2az$, whose Rado number was determined in Section 3.2 to be a^2 .

Conjecture 6.2. $R(x + a(x + y) = 2aw) = a^2$

Note that the lower bound is established by the theorem in Section 3.2.

7 Empirical Results

Tables of computed 2 and 3-color Rado numbers are can be found in the appendix. We include an extension of the table of computed 2-color Rado numbers presented in Meyers and Robertson's paper [5], as well as the first published results on 3-color Rado numbers.

In the appendix, we also include our formulation of the problem of determining Rado numbers as a Boolean Satisfiability problem - the motivation for our own algorithm which was loosely based off of a randomized SAT algorithm. It is our belief that utilizing powerful SAT solvers will provide a new route to the computation of Rado numbers for equations in a large number of variables and colors. It will be of interest to see if industrial SAT solvers will be able to make headway in this regard.

The present paper focuses on the efficient computation of Rado numbers, and presents several bounds on Rado numbers for a few classes of equations. A new proof of Rado's theorem is given, yielding better bounds on Rado numbers in certain cases. Finally, a probabilistic method of constructing lower bounds of Rado numbers is given. Significant progress was made in the area of computation of Rado numbers - the first comprehensive list of 3-color Rado numbers has been developed. Additionally, further exploration of our proof of Rado's theorem may give rise to new bounds. The probabilistic method bounds represent, to the best of our knowledge, the first expressions giving bounds on Rado numbers in greater than 2 colors.

References

- [1] W. Gasarch, C. Kruskal, and A. Parrish. Van der Waerden's theorem: Variants and applications. www.gasarch.edu/~gasarch/~vdw/vdw.html.

- [2] W. Gowers. A new proof of Szemerédi's theorem. *Geometric and Functional Analysis*, 11:465–588, 2001. <http://www.dpmms.cam.ac.uk/~wtg10/papers/html> or <http://www.springerlink.com>.
- [3] R. Graham, B. Rothchild, and J. Spencer. *Ramsey Theory*. Wiley, 1990.
- [4] B. Landmann and A. Robertson. *Ramsey Theory over the integers*. AMS, 2003.
- [5] Aaron Robertson and Kelen Meyers. Some Two Color Four Variable Rado Numbers. *Advances in Applied Mathematics*, 41:214–226, 2008.
- [6] Uwe Schöning. A probabilistic algorithm for k-SAT and constraint satisfaction problems. *Discrete Mathematics*, 197-198:397–407, 1999.
- [7] Heiko Harborth and Silke Maasberg. All 2-color Rado numbers for $a(x+y) = bz$. *Discrete Applied Math*, 95:279–284, 1999.
- [8] S. Shelah. Primitive recursive bounds for van der Waerden numbers. *Journal of the American Mathematical Society*, pages 683–697, 1988. <http://www.jstor.org/view/08940347/di963031/96p0024f/0>.
- [9] R. Rado. Studien zur Kombinatorik. *Math. Zeit.*, 36:242–280, 1933.
- [10] B. van der Waerden. Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wisk.*, 15:212–216, 1927.
- [11] I. Schur. Über die Kongruenz of $x^m + y^m \equiv z^m \pmod{p}$. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 25:114–116, 1916.
- [12] R. Rado. Notes on combinatorial analysis. *Proceedings of the London Mathematical Society*, 48:122–160, 1943.
- [13] J. Spencer. Asymptotic lower bounds for Ramsey functions. *Discrete Mathematics*, 20:69–76, 1977.
- [14] R. Graham, B. Rothschild, J. Spencer. *Ramsey Theory*. Wiley-Interscience, 1990.
- [15] K. Meyers, A. Robertson. Two Color Off-Diagonal Rado-Type Numbers *Electronic Journal of Combinatorics*, 13, 2007

A Algorithm

A.1 Our Algorithm

Previous attempts at the computation of Rado numbers have relied on backtracking methods - computationally inefficient in the analysis of Rado numbers in a large number of variables or colors. We developed a simple probabilistic algorithm to compute Rado numbers. Note that our algorithm is a variant of Schöning's algorithm for the Boolean satisfiability (SAT) problem. In the following section we elucidate the method of converting the problem computation of Rado numbers to the SAT problem. Briefly, our algorithm functions as follows.

Input equation: $a_1x_1 + a_2x_2 + \dots a_kx_k = 0, c$

Set $N = k$

RandomColor: Assigns a random color, in $[1, c]$, to each number in $[1, N]$.

FindSolutions: Returns integral solutions in $[1, N]$.

For 1 to $3N\sqrt{N}(\frac{4}{3})^N$

Solve: Iterates through each solution, finds the first monochromatic solution (x_1, x_2, \dots, x_j) . If no monochromatic solution is found, N is incremented, assigned a random color. Call *FindSolutions*.

ChangeColor: A number in $[1, j]$ is randomly chosen, and the color of x_j is randomly changed. *Solve* is repeated.

Return N

Due to the probabilistic nature of the solution, the algorithm is able to quickly find "bad" N 's, those containing colorings without monochromatic solutions, and move on. However, because the algorithm is probabilistic, there exists an error - or the probability of the algorithm missing a valid coloring. For the number of steps chosen in our algorithm, the error is bounded by e^{-20} (where e is the base of the natural logarithm). Consult Schöning's original paper [6] for the proof of this result.

This algorithm was inspired by the k -SAT algorithm developed by Schöning in 1999[6]. We note in the appendix how the problem of the computation of Rado numbers can be easily transformed to an instance of k -SAT.

For $c = 2$, our algorithm runs in $poly(n)(\frac{4}{3})^n$ time; however, for $c \geq 3$, note that the algorithm's running time is bounded by $(poly(n)(\frac{4}{3})^n)^c$.

In the appendix, we present in more detail the expression of the Rado problem as an instance of SAT. Although in the current paper no SAT Solvers were utilized, it is our belief that it will be possible to compute the Rado numbers of complex equations in a large number of colors with industrial SAT Solvers, utilizing the methods elucidated below.

B Computing Rado Numbers with SAT

B.0.1 Overview of the Boolean Satisfiability Problem

Given a boolean formula, the boolean satisfiability problem (SAT) is to determine the values of the boolean variables within the formula that will make the expression evaluate to True. The SAT problem was the first problem proved to be NP-Complete, and, as such, there are no known methods for efficiently solving SAT on a large scale. However, the SAT problem

is easy for small inputs, and many algorithms have been developed to decrease the running time of the algorithm. We introduce some notation: Conjunctive Normal Form (CNF) refers to a boolean expression with literals or their negations separated by OR's within clauses separated by AND's. An example is given below:

$$(a_1 \vee a_2 \vee a_3) \wedge (b_1 \vee \neg b_2 \vee b_3) \dots$$

The k -SAT problem takes a boolean expression in CNF form as its input, with at most k literals in each clause, and outputs an assignment of values to the literals, that evaluates the expression to TRUE. If such an assignment does not exist, the algorithm will output FALSE. Many problems have been reduced to instances of the k -SAT problem, and in the following sections we show how the problem of finding 2-color Rado numbers can be posed as an instance of k -SAT. By implementing efficient algorithms that have been developed for general case SAT problems, we believe it will be possible to greatly decrease the run time for the computation of Rado numbers.

B.1 Schöning's Algorithm and Rado as a SAT Problem

Recall that our algorithm was loosely based off Schöning's algorithm for the Boolean Satisfiability problem. Here we present an outline of Schöning's probabilistic SAT algorithm [6], and describe deviations from this algorithm in our implementation.

Briefly, the algorithm randomly assigns values to the Boolean variables present in the expression. It then finds the first clause in the expression that evaluates to false, and randomly picks a variable in the clause and changes its value. This process is repeated $3n\sqrt{n}(\frac{4}{3})^n$ times and if no solution to the SAT problem is solved, the algorithm returns UNSATISFIABLE.

In our algorithm, rather than considering an initial random coloring of $[1, N]$ after N is incremented, the previous coloring of $[1, N-1]$ that contained no monochromatic solutions is carried over. Clearly, many colorings that contain no monochromatic solutions in $[1, N-1]$ will also contain no monochromatic solutions in $[1, N]$, so this step reduces unnecessary searching in many cases.

Additionally, we have included a factor k into the function specifying the number of steps: $k(3N\sqrt{N}(\frac{4}{3})^N)$, which can be adjusted to give varying degrees of error from the probabilistic algorithm. Note that for $k = 1$, the error - or probability of missing a valid coloring, is given by e^{-20} . Consult Schöning's original paper [6] for the proof of this result.

B.1.1 Finding Two-Color Rado Numbers of Three Variable Equations with 3-SAT

Given an equation, let the solutions of the equation, in the interval $[1, N]$ be given by $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots (x_n, y_n, z_n)$. Let the colors be $\{0, 1\}$, and define the color of j to be the boolean variable C_j .

The below expression will evaluate to true if and only if the current coloring contains no monochromatic solutions:

$$(C_{x_1} \vee C_{y_1} \vee C_{z_1}) \wedge (\neg C_{x_1} \vee \neg C_{y_1} \vee \neg C_{z_1}) \wedge (C_{x_2} \vee C_{y_2} \vee C_{z_2}) \wedge (\neg C_{x_2} \vee \neg C_{y_2} \vee \neg C_{z_2}) \dots$$

To see why, imagine a monochromatic solution (x_i, y_i, z_i) . The color is either 0 or 1, so one of $(C_{x_i} \vee C_{y_i} \vee C_{z_i})$ and $(\neg C_{x_i} \vee \neg C_{y_i} \vee \neg C_{z_i})$ must evaluate to false. Because the clauses are in CNF, if one clause evaluates to false, the expression must as well.

B.1.2 Extension to the k -Variable Case

Just as 3-SAT can be used to compute the Rado number for an equation in three variables, the Rado number of a general equation in k variables can be computed with k -SAT. The expression for the 3-SAT case contains one literal for each variable in a clause. Extending this to the general case, we see that each clause should contain a literal for each variable in the equation. Below is an example, the particular solution is given by $(x_1, x_2, x_3, x_4, \dots, x_k)$.

$$(C_{x_1} \vee C_{x_2} \vee C_{x_3} \vee C_{x_4} \cdots \vee C_{x_k}) \wedge (\neg C_{x_1} \vee \neg C_{x_2} \vee \neg C_{x_3} \vee \neg C_{x_4} \cdots \vee \neg C_{x_k}) \cdots$$

B.1.3 Extension to the c -Color Case

We now show how k -SAT can be used to solve for the c -Color Rado number of an equation. Define $i_l = 1$ if $COL(i) = l$, for $0 \leq l \leq c - 1$. Clearly, only one i_j can equal 1 for a given coloring. Then, the following expression will evaluate to true if and only if there exists no monochromatic colorings under the current coloring. Let the solutions to the equation be given by $(x_1, x_2, x_3, \dots, x_k)$. Then, the following expression evaluates to true if and only if the current coloring does not contain any monochromatic solutions.

$$(\neg x_{1_1} \vee \neg x_{2_1} \vee \neg x_{3_1} \cdots \vee \neg x_{k_1}) \wedge (\neg x_{1_2} \vee \neg x_{2_2} \vee \neg x_{3_2} \cdots \vee \neg x_{k_2}) \cdots \wedge (\neg x_{1_c} \vee \neg x_{2_c} \vee \neg x_{3_c} \cdots \vee \neg x_{k_c})$$

C Computed Rado Numbers

In this section we present some of the Rado numbers that were computed using our algorithm. Note that due to the probabilistic nature of the algorithm, the probability of missing a valid coloring is bounded by e^{-20} . Consult Schöning's original paper [6] for the proof of this error bound.

C.1 3-Color Rado Numbers

Here we present a table of computed lower bounds for 3-Color Rado numbers of equations of the form $(a(x - y) = bz)$ using our algorithm- to the best of our knowledge no previous studies have presented such a table for $R_3(\mathcal{E})$.

$R_3(a(x - y) = bz)$	$a = 1$	2	3	4
$b = 1$	14	14	27	57
2	42	14	31	14
3	78	56	14	64
4	94	43	67	14
5	142	108	85	81
6	161	80	42	54
7	178	157	136	128
8	193	127	157	43
9	213	190	80	163
10	237	142	202	98
11	247	227	211	204
12	258	156	120	78
13	291	255	250	244
14	299	178	267	154
15	318	302	140	278
16	348	197	309	125
17	358	334	317	312
18	380	216	167	192
19	416	370	372	351
20	416	243	375	148
21	440	410	179	367
22	461	252	411	230
23	462	439	424	418
24	485	276	196	155
25	500	495	446	438

C.2 2-Color Rado Numbers

Here we present a table of computed 2-Color Rado numbers for equations of the form $2(x - y) + az = bw$ - an extension of some of the results in Meyers and Robertson's paper [5].

$2(x - y) + az = bw$	$a = 1$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$b = 1$	29	76	86	106	119	145	156	190	201	238	258	290	326	361	389	430
2	8	11	23	19	40	29	65	41	92	55	123	71	158	89	191	108
3	5	4	9	12	10	16	21	18	25	28	29	32	41	36	48	56
4	4	4	8	5	12	8	22	9	29	15	45	17	54	23	76	25
5	1	8	5	4	9	6	7	12	13	12	11	16	19	16	25	16
6	4	1	4	4	7	5	9	4	15	8	22	9	26	10	32	15
7	3	8	1	6	5	4	12	6	10	8	12	12	14	16	13	14
8	4	4	6	1	5	4	7	4	9	6	13	5	15	7	22	8
9	5	10	9	7	1	9	5	6	15	9	13	9	13	8	15	16
10	12	9	6	4	6	1	5	5	7	8	12	6	11	6	15	6
11	7	16	7	9	7	8	1	8	5	8	18	10	10	11	16	10
12	8	4	9	3	8	4	7	1	9	5	8	6	9	7	12	9
13	9	23	6	14	6	10	8	8	1	10	5	8	21	10	11	12
14	10	10	12	4	7	3	10	4	6	1	5	5	8	11	11	8
15	9	27	10	16	5	9	6	10	9	10	1	10	5	8	24	12
16	13	12	12	5	9	4	8	3	9	4	6	1	8	5	11	8
17	14	35	10	24	10	14	10	9	6	10	8	11	1	8	5	8
18	15	14	16	7	12	9	9	6	12	6	10	9	9	1	9	5
19	16	45	17	26	7	16	8	15	8	9	6	10	9	11	1	10
20	16	16	18	8	13	6	11	5	12	8	10	5	10	4	15	1
21	26	49	15	32	7	22	11	14	9	11	10	14	10	10	9	12
22	27	18	22	9	16	7	12	6	11	5	14	8	12	5	11	6
23	28	62	20	35	13	24	9	16	11	14	5	12	10	14	10	10
24	30	20	28	9	16	10	14	4	12	6	14	5	12	8	12	5
25	31	61	17	44	18	26	10	21	8	15	11	13	6	13	15	14
26	33	22	30	17	20	9	16	10	14	4	12	11	18	6	14	8
27	34	82	27	48	18	34	11	23	27	17	12	14	11	12	15	9
28	35	24	16	18	22	10	19	8	16	10	14	4	14	11	13	5
29	37	73	21	59	18	36	12	25	16	22	14	17	12	15	11	12
30	38	26	39	20	28	9	19	11	18	11	16	9	14	7	20	10

Continued table of computed lower bounds for 2-color Rado numbers of equations of the form $2(x - y) + az = bw$.

$2(x - y) + az = bw$	$a = 1$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$b = 31$	44	102	27	63	22	45	11	32	10	24	16	19	15	19	12	20
32	45	38	46	21	29	13	20	10	20	11	14	10	16	10	16	7
33	47	103	24	68	19	46	12	34	18	24	17	23	15	22	15	18
34	48	40	44	23	31	14	22	11	18	11	20	11	16	12	16	10
35	49	108	25	74	24	51	14	36	12	25	19	24	17	21	15	21
36	58	43	57	24	32	15	27	9	23	13	18	15	20	12	18	9
37	60	116	32	93	21	60	14	46	20	32	12	24	18	24	17	21
38	62	48	66	26	38	16	28	13	24	12	18	13	16	13	16	11
39	63	129	57	86	36	63	21	46	18	34	15	25	26	23	16	23
40	64	50	58	27	42	16	30	14	24	14	19	12	22	13	20	12
41	74	137	48	84	25	74	23	50	15	36	18	32	21	27	20	26
42	76	53	72	32	48	26	32	14	28	10	21	16	21	14	20	16
43	78	144	56	119	39	78	24	58	16	44	15	34	16	27	22	27
44	80	59	82	33	50	27	32	15	29	14	28	10	24	15	19	17
45	81	152	52	97	25	81	27	54	18	46	18	34	17	30	45	29
46	102	62	73	35	57	28	38	16	30	15	29	12	22	10	20	15
47	104	193	54	124	42	79	29	63	22	48	17	36	20	32	19	27
48	106	64	96	36	64	30	44	17	30	14	28	9	23	18	24	16
49	108	201	57	135	34	89	36	74	19	50	20	43	12	35	18	29
50	111	84	108	44	54	31	46	21	32	17	32	16	25	9	28	18
51	115	213	62	140	41	86	33	77	21	60	24	45	20	35	21	33
52	117	87	90	46	70	33	48	22	32	15	34	13	27	16	24	9
53	120	224	64	146	42	104	35	80	21	54	22	47	20	37	20	34
54	122	91	121	48	72	34	50	23	38	16	30	18	29	11	26	19
55	124	235	71	151	50	113	31	92	22	56	22	49	22	44	24	36
56	135	98	116	49	82	35	56	24	44	16	30	14	28	18	27	12
57	137	246	74	147	45	118	34	81	23	72	33	58	21	46	21	35
58	140	102	108	66	91	37	64	25	46	21	32	15	33	19	29	17
59	142	257	77	159	47	109	39	92	24	56	22	60	26	46	20	36
60	144	105	129	68	86	38	66	25	47	22	34	15	35	14	31	18